THE PREPARATION OF THE HIGH SCHOOL TEACHER OF MATHEMATICS

THERE is an important sense in which the preparation of every teacher is beyond the reach of human influences. His preparation has begun before his birth. He is either endowed by nature with personal qualities that should forever exclude him from the ranks of the teaching profession, or he possesses such qualities, as under appropriate training, enable him to overcome the inevitable difficulties that will beset his path, and ultimately to attain varying degrees of usefulness from mediocrity to the highest skill in his art. If he is a physical or mental weakling, if he is stolid and heavy, if he is indifferent to nature, human nature, and art, if he lacks enthusiasm in the pursuit of his subject and never feels the glow of conscious mastery, if he has a crabbed or irritable disposition, if he is brilliant but unsympathetic, if he lacks an interest in his pupil at least equal to his interest in his subject, if he has no tact, and is lacking in the sense of humor that often furnishes the silver lining to an otherwise black cloud of youthful idleness or seeming perversity —in a word, if he is not physically and mentally vigorous, alert and active, if he is not interestedly and healthily responsive to the varied interests of life, if he cannot cherish a feeling of good will and maintain a hopeful and encouraging attitude in spite of many discouragements and some failures, whatever he may be able to achieve in other callings, he ought never to be a teacher. I need hardly say that, in what follows, adequate natural capacity and a responsive nature are assumed, and that my discussion pertains only to the preparation of the would-be teacher who possesses these characteristics.

The preparation needed by every high-school teacher is both general and special; that is to say, it should cover the essential elements of a liberal education, and special training in that sub-

ject or group of closely related subjects which the teacher expects to teach; together with enough professional training to show him his responsibilities to his pupils as well as to his specialty, and help him to become as good a teacher as possible as soon as possible. That a high-school teacher should, in general, have profited by an education at least equivalent to that afforded by a good American college, ought to be universally recognized. Since this is not the case, one must assume either indifference to or ignorance of the importance of a liberal culture for high-school teachers on the part of those who employ the teachers, or are responsible for their employment. Under such circumstances, it becomes one's duty to do what he can toward influencing public opinion on this very important matter. I am aware that in this presence this endeavor is unnecessary, if not somewhat presumptuous; but I am sure that the deliberations of this club are intended not merely for the enlightenment of its members but also to promote the dissemination of wholesome educational ideals in the larger community outside, which the members of this club serve as educational advisers and teachers, and the intelligent coöperation of which they aim to secure in every legitimate way. It seems worth while, therefore, to point out briefly the serious consequences of indifference or ignorance on the part of employers of high-school teachers as regards the general culture essential to real efficiency.

It must be borne in mind that high-school pupils are no longer little children, uninformed and unsophisticated. Besides possessing a considerable store of general information, they have usually learned to read human nature tolerably well where their own interests are concerned. Superficial knowledge, and limited mental power, narrow views of life, rusticity of manner —all of them marks of meager culture—rarely escape detection in a high school; and particularly for the brighter and socially superior pupils offer a serious obstacle to the teacher's usefulness, if they do not destroy it altogether. It is a serious disadvantage to every high-school pupil, whether he is aware of it or not, perhaps even more serious if he does not know it than if

he does, to have his mental horizon determined by the narrow mental horizon of his teacher; his intellectual vistas and sympathies limited or dwarfed by the inadequate intellectual insight and want of perspective on the part of his teacher; to have his notions of social refinement and cultivation unformed, or deformed, or even perverted by the uncultivated man or woman who happens to be his teacher. The high school supplies to most pupils their last chance at these stores of inspiration and guidance, and they should be the very best. Such disadvantages to the pupil do not appear, and such obstacles to the teacher's success can hardly exist when the general culture and refinement of the high-school teachers are sufficient always to command the just respect and challenge the regard, if not to inspire the imitation, of the best pupils. For such an equipment those who have tried it will agree, I am sure, that, in general, four years of training in a good college are little enough.

This view is strengthened by the reflection that there is no period in a young person's life in which impressions received produce a more lasting effect, in which incipient interests, and habits of thought and conduct are more permanently influenced than during the period covered wholly or in part by secondary education—the period of adolescence. It is often said that the earliest impressions are the most lasting and the earliest training is the most important for intellectual and moral development, and for the future usefulness and happiness of the individual; but I cannot believe that in most instances this is a true statement of the case, so far, at least, as those pupils are concerned whose school career is continued into and through the high school. If good early training were always followed by equally good subsequent training, if the child's opportunities for growth in knowledge and power were continuous, if his moral training and his social environment improved with his growth from early childhood through later childhood and youth, if his earlier acquisitions were really made to serve continuously and progressively as the foundation for continuous subsequent growth, then the earlier and earliest training would be of the utmost

importance; for it would be the foundation on which later development would be most economically and securely laid. But such conditions of development are rare. The cases are not common in which each stage of a child's progress is so nicely adjusted to the previous one. Such an arrangement of our courses of study and our teaching processes is as yet too commonly rather a vaguely conceived ideal. Moreover, the instability of the population, the perpetual migration of people in this country from one place to another, enormously increases the difficulty of approximating to such an ideal, even when it is consciously and conscientiously aimed at. But suppose that such an ideal were generally realized. It would still remain true that childhood in its first dozen years or so, with its ready adaptability to changing conditions, its rapidly changing dominant interests, in a word, its instability, would be affected most strongly by its latest influences. In my opinion it is quite possible for the later school life to make good the defects of early training; or, it may undo very largely what has been well begun, on the one hand, as, on the other, it may build on an excellent earlier foundation a superior superstructure. If this be true, and I believe it is (in the absence of scientific knowledge one can only generalize from his own experience and observation), the great importance of good teaching, wise management, and the most wholesome and refining general atmosphere in the high school are apparent. In any event, it will be admitted, I think, that as the period of adolescence approaches, and especially during that period, the instability above referred to rapidly diminishes. The individual gradually emerges. The child becomes a youth. This is a critical period in the life of every human being. To assume the wise guidance of young people during this important period is the exalted function of the secondary school-teacher. He is to be the inspiring, sympathetic discriminating, and vigorous guide and leader of boys and girls just developing into manhood and womanhood. To be such a guide and leader in very truth he must have resources, both natural and acquired. The least that should be demanded of him is that he shall have taken pains to secure an equipment of knowledge that

will give him broad, sane, and healthy views of life, with its duties and its privileges, and liberal intellectual sympathies; together with a conscious power in some one field that enables him to maintain, both for himself and his pupils, a high standard of achievement. Such an equipment every college graduate who has made good use of his opportunities may possess. It is not often that one who has not profited by such an education can be expected to possess it.

There are special reasons why the teacher of mathematics should have a liberal equipment of general culture in addition to special knowledge and power in his particular subject. Pure mathematics is profound and interesting; but its subject-matter, save in its elements, is so remote from the common interests of men that its devotees are in constant danger of what may be called a professional or academic isolation; and this isolation is almost sure to increase with increasing devotion to the subject. It has fewer points of contact with the ordinary affairs of life than natural science, or language, or history; to say nothing of subjects like economics and political science. In this respect mathematics differs from all other subjects. Factoring, radicals, and quadratic equations, polygons, paralleropipeds, and spherical triangles, sines, tangents and trigonometric formulæ, the theory of equations, determinants, and complex numbers, point and line coördinates, involutes and evolutes, derivatives, differential equations, and elliptic functions, may completely shut out from view the living panorama of nature and society in which most men live and move, and have their being, and from which the young, in particular, derive most of their incentives and conscious purposes. In my opinion it is therefore not asking too much that the high-school teacher of mathematics should possess and understand the importance of the general training that enables him to appreciate extra-vocational, *i. e.*, for him extra-mathematical interests. It ought never to be possible for the teacher of mathematics, however high he may rate the importance of his own subject and its beneficial effect on the pupil, to become one who measures the capacity of all pupils solely by

their ability to "do sums and to work problems," and the goodness of any course of study chiefly by the work in mathematics it prescribes.

But after all has been said that can be said of the necessity of broad general culture for the teacher of mathematics it is still emphatically true that his efficiency depends ultimately on his special training, on his resources in and power over his own subject; for it is through that subject that his special duty to the pupil is to be done. Either the pupil is to receive through him a peculiar insight into the marvelous system of the external world, some comprehension of the wonderful power and fertility of the human mind in one of its fields of activity, a quickening of his intellectual life through the knowledge and insight, the clearness and adequacy of exposition of mathematical truths characteristic of a scholarly, enthusiastic, well-trained teacher; or he must forever remain indifferent to these interesting, and some minds fascinating experiences, because both he and his teacher move about vaguely in a "world unrealized," where the intellectual fog never rises, and where the road traversed today must be re-traversed with the same dim uncertainty tomorrow.

If the teacher of mathematics has little mathematical ability, and if at the same time he has not had sufficient training, he is almost sure to carry on his work with a benumbing inadequacy of comprehension and exposition which soon becomes chronic, and through which the pupils come to look on mathematics as a highly artificial subject of little real interest or practical utility; a subject in which success does not depend on common sense and patient study, but on a certain inborn ingenuity of manipulating postulates, hypotheses, and previous propositions, and in which absurdities are as valid as realities.

A certain college student of my acquaintance must have had this kind of instruction. She said she had studied algebra before coming to college, but she had become interested in the subject only when she took up equations and proved "things." Whereupon I asked her if she had ever seen the paradox by which, through a series of equations, any number may apparently be

proved equal to nothing. As she had never seen it, I showed it to her, securing her assent to the several steps as I went on. When we arrived at the conclusion, $2=0$, I handed her the paper and looked up with the half apologetic, half triumphant manner of one who expects after a very brief triumph and a rather lame defense to yield the point in question. But nothing of the sort happened. She merely said "That's all right" and handed the paper back to me. "But," I said, "how can it be? Two cannot be really equal to nothing." "O," she said, "that's algebra."

The situation is not much improved, although it is quite different, if the teacher has good ability, but insufficient training. In that case he is oppressed by the consciousness of the heavy demands made on him and of his inability for a long time to respond to them as he should. The work must be done somehow, the classes cannot wait. He is thus obliged to carry on a discouraging struggle against tremendous odds. Meanwhile, his pupils are the losers, and the reputation of the subject suffers. Similar statements could, of course, be made with respect to inadequate preparation in other subjects as well as mathematics, but the immediate consequences are not so conspicuous. The definiteness and rigor of mathematical reasoning afford a constant and ruthless exposition of the teacher's shortcomings—an exposition which in other subjects is neither so glaring nor usually so disastrous. The teacher of mathematics must be a logical reasoner and ready in manipulation. If his training has left him without these powers, his other mental powers will avail him nothing. Either the pupil is right or he is wrong. Neither teacher nor pupil can escape the consequences of false reasoning or lack of skill in handling mathematical expressions. So, too, the glimpse the teacher gets of fields unexplored, which he vaguely realizes must have an influence on the interpretation of the work on which he is engaged, is a constant intimation of inadequacy, and so a source of self-accusation that heightens the acute "misery of conscious weakness" which he is sure to feel, and which is one of the most paralyzing of all the untoward influences that oppress the conscientious but meagerly equipped

teacher. There is no heavier burden than the burden of accepted duties that one feels he cannot adequately perform.

There is, of course, always hope for the able teacher inadequately prepared, for he may, by dint of hard work, ultimately achieve at least a moderate efficiency, although at the expense of many pupils; but there is no hope for the ignorant teacher of poor ability unconscious of his own ignorance. In either case the want of adequate preparation before actual service begins casts its shadow over his entire professional career.

To teach mathematics well in the high school, it is, therefore, hardly necessary to argue that one must have a thorough knowledge of the subject, a knowledge that is far in advance of pre-collegiate study, *i. e.*, far in advance of a good acquaintance with the branches of mathematics usually found in the high-school curriculum or such a presentation of them as is contained in the usual text-books. It is hardly necessary to argue that with an equipment limited to pre-collegiate study, the teacher of elementary mathematics is unable to comprehend the relative importance of the different phases of his subject. He may, and usually does, neglect important aspects and magnifies trifles. He treats facts and processes as ultimate ends in themselves, instead of means to ends. He never gets the comprehensive point of view from which the subject is unified and gains full significance in his own and the pupils' minds. His pupils not infrequently learn many things which subsequently must be unlearned—an expensive and exasperating experience. That he may escape this unfortunate situation, that he may from the start enter on his work well equipped for the demands that are to be made on him, I purpose now to enquire what should a good course of study, to be pursued by the high-school teacher of mathematics as special preparation for his work, comprise?

Bearing in mind that this course of study should enable the teacher to appreciate the relative importance of the different phases of his subject and of their interdependence throughout, and so enable him to select with certainty and wisdom those portions of mathematics essential to the elements of a liberal

education, or for future specialization in mathematics, if the pupil's interest and probable career should lead him into that field; and also that the teacher should know and be able to point out the significance of mathematics for the adequate development of power over other sciences, it is clear that the teacher's preparation must cover both pure and applied mathematics, and a general training in elementary physical science. Only through such knowledge is it possible to expect confidently that the teacher shall have an adequate, a wise, and firm grasp of essentials; and be able to impart to the pupil the scientific interest born of insight, unified knowledge, and perspective; and also a conviction of the fundamental utility of his subject in the pursuit of other sciences and in the practical affairs of life.

In view of these considerations, I suggest the following college course of study for the equipment of the high-school teacher of mathematics:

Advanced algebra, about fifty lessons, theory of equations, about fifty lessons (together with determinants and complex numbers); solid geometry, about fifty lessons; trigonometry, plane and spherical, about fifty lessons; surveying, about fifty lessons; calculus, about one hundred lessons; mechanics, about one hundred lessons; history of mathematics, about fifty lessons; physics, about one hundred lessons (general, with quantitative laboratory work); astronomy, about fifty lessons; chemistry, including mineralogy, about one hundred lessons; application of the calculus to light and heat, or to electricity, about one hundred lessons; history and theory of education, about one hundred lessons; methods, about fifty lessons.

Such a course of study should give the teacher of mathematics the original awakening and impetus, and furnish the permanent sources of inspiration and guidance without which growth and high efficiency are impossible. It places the teacher at the center of his subject, whence he can see and use its resources to the best advantage. The command it gives him over the resources, the problems, the historical development, and the practical and theoretical utility of elementary mathematics may

be expected to enrich and vitalize his teaching. Under a teacher thus equipped, the pupils feel that they are dealing with a fertile and a beautiful subject, and that they have a steady, able, and sure guide to the mastery of it. The halting and labored procedure, so characteristic of one who is not master of the situation, will be wanting; the pupils will perceive a definiteness and directness of aim and procedure, a facility of handling mathematical expressions which shows them how real it all is, how important, interesting, and useful it all is, and how accessible it all is to patient study; and, to the mathematical spirits among them, how fascinating and how simple it all may become. Such a course of study will make it impossible for the teacher of elementary mathematics to regard high-school algebra, geometry, and trigonometry as "higher mathematics," as I have heard some teachers call these subjects; but he will nevertheless have a just and high regard for these subjects, and will address himself to teaching them accordingly.

The teacher who has taken such a course of study will not be likely to teach nonsense about negative numbers or about imaginaries in elementary algebra. What he does teach about them the pupils will not have to unlearn at some future time. He will not believe that he has done his duty when he has led the pupil to solve simple equations with numerical coefficients, and allowed them to regard the literal equation as exceptional, and therefore entitled to only passing notice; he will not feel that the last word has been said on the significance of systems of equations when the pupils have learned the three common processes of elimination and have applied their knowledge to solving the usual highly artificial problems of the elementary algebras; he will not be able to regard the solution and discussion of the quadratic equation as virtually the end of achievement in algebraic analysis, and so on. So in geometry, he cannot be satisfied with exactness of language only; he must have good logic as well; he will attach due importance to the mastery of the propositions proved and the problems solved in the textbook; but he will never feel that the mastery of these proposi-

tions, with only an occasional "extra," gives pupils the comprehension and power they ought to gain from the subject, and so on. He will have learned that the branches of elementary mathematics have important interrelations, and will provide for a proper recognition of these interrelations in his teaching, thereby enhancing the interest and the comprehension of each. These are commonplaces, I know; my excuse for laying stress on them is that their significance is too often disregarded in practice.

That analytical geometry should form a part of this course of study is obvious. The knowledge of the conic sections which it affords and the great extension it gives to earlier conceptions of geometry, are so important, the relations between algebra and geometry which it reveals are so interesting and valuable, and the facility in handling mathematical expressions which it cultivates is so useful, that every lover of mathematics would insist on a training in analytics, and every teacher should possess this training for its influence on his teaching.

Similarly, little argument is needed to justify the appearance of the calculus and its applications in this course of study. The calculus is of such value to the mathematician as an instrument of research and as a basis of other sciences, that for him it is simply indispensable. For the teacher of elementary mathematics it throws such a flood of light on principles and extends their significance, both in pure and applied mathematics, and develops the mathematical skill already acquired to such an extent that it is of inestimable value in helping the teacher to develop comprehension, interest, and skill on the part of his pupils. To illustrate the influence of the calculus on the equipment of the teacher I need only refer to its effect on his power over the theory of limits, to the insight afforded by the applications of the calculus to analytics, to the facility of manipulation cultivated from the very beginning in the processes of differentiation and integration, and in the applications of the calculus to analytics already referred to, and to the appreciation the calculus affords of the utility of mathematics through its applications to mechanics, and to the study of light, heat, or electricity.

This course of study provides also for an acquaintance with the history of mathematics. It is strange that this interesting subject should have been neglected so long in this country. To this day I know of no college or university among us that offers a course in the history of mathematics. This is all the more strange when we think of the addition to the teacher's resources comprised in a knowledge of this subject. Quite apart from the clearer insight which such a knowledge gives into principles and processes — which is, of course, important — it should be borne in mind that the history of a subject is a part of the history of culture, and so an important part of the history of civilization. The inspiring victories of peace, the intellectual and moral conquests of the race, has for most minds at least as high an interest as the victories of war. The history of brilliant discoveries, or of the patient endeavor by which men have slowly worked out a science or a system of thought may therefore be employed to introduce into the teaching of every science a human element of great value; and especially into a science which, like mathematics, is necessarily formal and abstract. When the discoverer of a proposition is named and some of the circumstances of his career are added, a wholly new interest is added to the geometrical truth which before was known only as abstracted from all human concerns — it existed only in the book; now it is recognized as belonging to life.

So, too, the physical sciences — astronomy, physics, chemistry, and mineralogy — add so much to the significance of pure mathematics, and contribute such teaching resources that no mathematician and no teacher would wish to dispense with a knowledge of them. I have therefore included these sciences in the course of study needed by the teacher of mathematics. Finally, every science has an important practical utility which justly adds to the value attached to it by its devotion and by the world in general. An appeal to this practical utility is of constant use in teaching, because it is sure to be appreciated by the pupils, to say nothing (when occasion requires it) by their parents. I have therefore added astronomy, surveying, and intro-

duction to navigation to the course of study; these subjects comprise those important practical applications of elementary mathematics which the mathematician needs to know, and which everyone can appreciate.

Thus far I have been dwelling on the scholarship required for the teacher's outfit in mathematics. This is, of course, because scholarship is fundamental. But it is obvious that scholarship and the power to use scholarship effectively in teaching are two very different things. Hence, the teacher should have a general professional training, including, but by no means limited to, instruction in method. The professional training should cover enough of the theory and history of education to determine from the very beginning a professional interest and atti tude on the part of the teacher in addition to his interest in his subject. It should enable him once for all to attain the conception that education is a rationalized endeavor, whose aim is the appropriate development of a human being, and his adaptation to the civilization into which he is born; and whose means are general culture, and in his hands the science of mathematics in particular. It should include also some instruction in methods of teaching mathematics. Prior to practice, it is not profitable, in my opinion, to spend much time on the details of teaching any subject. The most valuable instruction in the details of method of teaching any subject is that which proceeds *pari passu* with practice. Such instruction is, in most cases, impossible. What, then, should the instruction in methods comprise?

The student's course in educational theory should have given him a conception of the educational value of mathematics, *i. e.*, what mathematics may do for the pupil, and what by no possibility can be got from it. Instruction in methods should make sure that such a conception of the educational value of mathematics defines the teacher's ultimate aim, the aim that underlies and permeates all his work. Such an aim is essential to an adequate treatment of the whole subject, and it relegates all the phases of the subject to their proper places, and assigns to each phase its proper emphasis; and it makes clear the importance of

bringing home to the pupil the interdependence of the several branches. Instruction in method should also make clear the importance of a definite aim or definite aims in each recitation, *i. e., of a certain point or certain points to be made in today's work*, and of the importance of permitting and requiring the pupils to make these points themselves, so far as possible; and finally it should suggest a few modes of procedure—devices of class management and presentation, sufficient to serve the young teacher as a basis for success in his first lessons, and for working out his own devices as fast as he needs them. For such instruction about fifty exercises divided as occasion may render desirable between algebra, geometry, and trigonometry, are enough. To this instruction should be added as much study as possible of the methods of one or more good teachers in their class rooms.

Of course, the teacher's preparation for his work cannot be completed once for all prior to service. That the teacher must continue to be a learner if he would win and maintain a high efficiency, is eminently true of the teacher of mathematics. The nature of the subject is such that it easily lends itself to routine treatment. But such treatment means death to efficiency. The teacher's intellectual sympathies must be kept alive by continual efforts to overcome difficulties similar to those experienced by his pupils. The teacher who ceases to work out proofs or solutions, and is content with his past acquisitions and his collections of solved problems has stopped growing, and ought to be retired.

It is hardly necessary to add that every teacher should be a reader of our three or four leading educational journals, and that the teacher of mathematics cannot afford to dispense with the reading of two or three of the current mathematical journals. Of course few teachers would find the whole of each issue of these journals either interesting or profitable. But there is enough in each issue to repay careful attention. They extend the teacher's professional horizon and keep him abreast of the times in educational movements; and the mathematical journals promote his

growth in his own subject by acquainting him from issue to issue with the progress his subject is making through the achievements of his fellow workers. Moreover, both the non-mathematical and the mathematical journals constitute a source of valuable information concerning the respective merits of teaching resources, new and old—the text-books or other books on this subject; and he will find frequent discussions of aims and methods of presentation that serve as a constant means of clarifying his own views and improving his own methods—all this, of course, whether he agrees with the opinions expressed in a paper on the teaching of mathematics, or in some book review, or not. These sources of information and inspiration are used far too little. The present generation of high-school teachers, with the exceptions of the principals, is not given to much reading of educational literature, even the best.

In this paper I have endeavored to describe briefly what seems to me a good preparation for the high-school teacher of mathematics. That this preparation does not imply an excessive demand is illustrated by the fact that at least one normal school, the state normal school of this state (Michigan), provides a special preparation for would-be high-school teachers of mathematics, that very nearly covers the course in mathematics which I have described. Indeed, my own views of what should be demanded of high-school teachers of mathematics in the way of preparation were strengthened and confirmed when I learned what the students at this normal school who wished to become high-school teachers of mathematics were obliged to take; and that about twenty-five students were actually pursuing the greater part of this work each year.

That I have described a preparation that most high-school teachers at present in service have not enjoyed, I know. That it is exceedingly important that such equipment should be increasingly insisted on before the teacher is permitted to begin his work I firmly believe.

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